MATCHING CONTROL LAWS FOR A BALL AND BEAM SYSTEM

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Abstract: This note describes a method for generating an infinite-dimensional family of nonlinear control laws for underactuated systems. For a ball and beam system, the entire family is found explicitly.

Keywords: Nonlinear control, mechanical systems

1. THE MATCHING CONDITION

This note presents an application of the method developed by Auckly, *et al.* (2000), to stabilization of a ball and beam system. The results are fully described in (Andreev, *et al.*, (2000), Auckly, Kapitanski (2000), and Auckly, *et al.* (2000)). An experimental comparison of a linear control law versus the nonlinear control laws described here will be given in the full paper, (Andreev, *et al.*, (2000)).

Let Q denote a configuration space. Let $g \in \Gamma(T^*Q \otimes T^*Q)$ be a metric. Let $c, f : TQ \to TQ$ be fiber-preserving maps. We assume that $c(-X) = -c(X)$. Let $V : Q \to \boldsymbol{R}$. The differential equation that we consider is

$$\nabla_{\dot\gamma}\dot\gamma + c(\dot\gamma) + \ grad_\gamma V = f(\dot\gamma). \tag{1}$$

Let $P \in \Gamma(T^*Q \otimes TQ)$ be a g-orthogonal projection. We consider the situation where a constraint $P(f) = 0$ is imposed. A system is called underactuated if $P \neq 0$.

Several recent papers propose to find control inputs so that the closed-loop system (1) would have a natural candidate for a Lyapunov function (Bloch, *et al.* (1998), Hamberg (1999), and van der Shaft (1986)). Auckly, *et al.* (2000) introduced the following matching condition and characterization of matching in terms of linear partial differential equations. A control input, f, satisfies the matching condition if there are functions $\widehat{g}$, $\widehat{c}$, and $\widehat{V}$ so that the closed loop equations take the form:

$$\widehat{\nabla}_{\dot\gamma}\dot\gamma + \widehat{c}(\dot\gamma) + \ \widehat{grad}_\gamma \widehat{V} = 0. \tag{2}$$

The motivation for this method is that $\widehat{H} = \frac{1}{2}\widehat{g}(\dot\gamma,\dot\gamma) + \widehat{V}(\gamma)$ is a natural candidate for a Lyapunov function because $d\widehat{H}/dt = -\widehat{g}(\widehat{c}(\dot\gamma),\dot\gamma)$. A straightforward computation shows that, the matching condition is satisfied if and only if

$$P(\nabla_X X - \widehat{\nabla}_X X) = 0, \tag{3}$$

$$\begin{aligned} &P(grad_\gamma V - \ \widehat{grad}_\gamma \widehat{V}) = 0, \\ &P(c(X) - \widehat{c}(X)) = 0. \end{aligned} \tag{4}$$

Equation (3) is a system of non-linear first order PDE's for $\widehat{g}$. It is perhaps surprising and pleasing that all of the solutions to (3), (4) may be obtained by first solving one first order linear system of PDE's and then solving a second set of linear PDE's. This is accomplished by introducing a new variable, λ, by $g(X,Y) = \widehat{g}(\lambda X, Y)$.

Theorem 1 *The metric, $\widehat{g}$, satisfies (3) if and only if λ and $\widehat{g}$ satisfy*

$$\nabla g\lambda\big|_{Im\ P^{\otimes 2}} = 0, \quad L_{\lambda PX}\widehat{g} = L_{PX}g. \tag{5}$$

In the special case of a system with two degrees of freedom, it is possible to write out the general solution to this set of differential equations. Following Auckly, Kapitanski (2000), express the underactuaded subspace as the span of a unit length vectorfield, PX. Choose coordinates x^1, x^2 so that $PX = \frac{\partial}{\partial x^1}$, and write $\lambda PX = \sigma\frac{\partial}{\partial x^1} + \mu\frac{\partial}{\partial x^2}$. For the λ-equation, (5), to be consistent the following compatibility condition must hold: $\partial([11,2]\,\mu)/\partial x^2 = \partial([12,2]\,\mu)/\partial x^1$. Starting with this equation and working backwards, all of the equations may be solved via the method of characteristics.

2. THE BALL AND BEAM SYSTEM

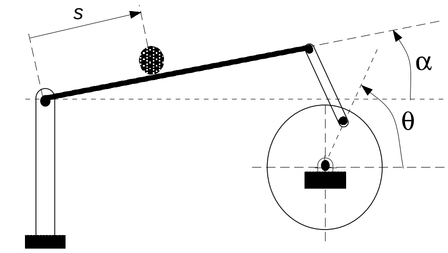

Fig.1. Nonlinear mechanical system.

As an application of our method consider the stabilization problem for the ball and beam system

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[1]This work was partially supported by NSF Grant No. CMS-9813182.

described schematically in figure 1. One can express α as an explicit function of θ. After rescaling, the kinetic energy of the system is given by:

$$T = \frac{1}{2}\dot{s}^2 + \alpha'\dot{s}\dot{\theta} + \frac{1}{2}\left(a_4 + \left(a_3 + \frac{5}{2}s^2\right)(\alpha')^2\right)\dot{\theta}^2$$

and $V = a_5 \sin(\theta) + (s + a_6)\sin(\alpha)$, where the a_k are dimentionless parameters. The projection, $P = (ds + \alpha' d\theta) \otimes \partial/\partial s$, so the control input u is related to f in (1) by $f = (ud\theta)^\sharp$. The resulting equations of motion are

$$\ddot{s} + \alpha'\ddot{\theta} + (\alpha'' - \frac{5}{2}s\alpha'^2)\dot{\theta}^2 + \sin(\alpha) = 0$$

$$\begin{aligned}\alpha'\ddot{s} + [a_4 + (a_3 + \tfrac{5}{2}s^2)\alpha'^2]\ddot{\theta} + 5\alpha'^2 s\dot{s}\dot{\theta} \\ +(a_3 + \tfrac{5}{2}s^2)\alpha'\alpha''\dot{\theta}^2 + a_5\cos\theta \\ +(a_6 + s)\cos(\alpha)\,\alpha' + a_7\dot{\theta} = u\,,\end{aligned}$$

where a_7 corresponds to inherent dissipation.

The general solution to the matching equations is

$$\widehat{g}_{11}(s,\theta) = \psi^2(\alpha)\,(h(y(s,\theta)) + 10\int_0^\alpha \frac{d\varphi}{\mu_1'(\varphi)\psi^2(\varphi)})$$

$$\widehat{g}_{12} = \frac{1}{\mu}(g_{11} - \sigma\widehat{g}_{11}), \quad \widehat{g}_{22} = \frac{1}{\mu}(g_{12} - \sigma\widehat{g}_{12}),$$

$$\widehat{V}(s,\theta) = w(y) \,+\, 5\,(y + s_0)\,\int_0^\alpha \tfrac{\sin(\varphi)}{\mu_1'(\varphi)\psi(\varphi)}\,d\varphi$$

$$-\,5\,\int_0^\alpha \tfrac{\sin(\varphi)}{\mu_1'(\varphi)\psi(\varphi)}\,\int_0^\varphi \psi(\tau)\,d\tau\;d\varphi,$$

where $y = \psi(\alpha)s - s_0 + \int_0^\alpha \psi(\tau)\,d\tau$, $\psi(\alpha) = \exp\{-5\int_0^\alpha \frac{\mu_1(\kappa)}{\mu_1'(\kappa)}\,d\kappa\}$, $\mu(s,\theta) = \frac{\mu_1'(\alpha)}{5s\,\alpha'}$, $\sigma(s,\theta) = \mu_1(\alpha) - \frac{1}{5s}\,\mu_1'(\alpha)$ and μ_1, h, and w are arbitrary functions. Also, $\widehat{c}^1 = -\alpha'\widehat{c}^2$, where $\widehat{c}^2(s,\theta,\dot{s},\dot{\theta})$ is an arbitrary function which is odd in $\dot{s}$ and $\dot{\theta}$. The final nonlinear control law is $u = u_g + u_V + u_c$, where $u_g = g(\nabla_{\dot{\gamma}}\dot{\gamma} - \widehat{\nabla}_{\dot{\gamma}}\dot{\gamma}, \frac{\partial}{\partial\theta})$, $u_V = \frac{\partial V}{\partial\theta} - g(\widehat{grad}_\gamma \widehat{V}, \frac{\partial}{\partial\theta})$, and $u_c = a_7\,\dot{\theta} - g(\widehat{c}(\dot{\gamma}), \frac{\partial}{\partial\theta})$. Using $\widehat{H}$ as a Lyapunov function, we obtain the following conditions that guarantee local asymptotic stability of the equilibrium: $\det(\widehat{g}(0)) > 0$, $\mathrm{tr}(\widehat{g}(0)) > 0$, $\det(\widehat{gc}(0)) > 0$, $\mathrm{tr}(\widehat{gc}(0)) > 0$, $\det(D^2\widehat{V}(0)) > 0$, and $\mathrm{tr}(D^2\widehat{V}(0)) > 0$,

Another way to check local asymptotic stability is to find the poles of the linearized closed-loop system. It is a theorem (Andreev, *et al.* (2000), Auckly, Kapitanski (2000)) that any linear full state feedback control law can be obtained as a linearization of some control law in our family.

A good stabilizing control law will produce a large basin of attraction, send solutions to the equilibrium in a short period of time, and will require little control effort. It is, unfortunately, not clear how to quantify these goals.

We have done some numerical simulation of various control laws in our family. We always pick the arbitrary functions in our nonlinear control law in such a way that the linearization at the desired equilibrium, $u_{lin} = a_8 + K_{bp}(s - s_0) + K_{ap}\theta + K_{bd}\dot{s} + K_{ad}\dot{\theta}$, is exactly the linear state feedback control law provided by the manufacturer of a commercially available system (Apkarian, (1994)). The numerical and experimental response of the system to various initial conditions will be recorded in the full version of the paper.

3. CONCLUSION

We believe that nonlinear control laws have the potential to achieve better performance than linear control laws. There are, however, several subtle questions which must be resolved before nonlinear control laws may be fully exploited in practice. The first question is how to quantify performance. The second question is how to pick a control law which will come close to optimizing performance. One interesting idea is to restrict attention to a class of control laws which generate a closed loop system of a special form. The hope is then that it will be easier to quantify the performance of such systems. We have shown that, in many situations it is possible to find all control laws which will result in a closed loop system of the form (2).

REFERENCES

Andreev, F., D. Auckly, L. Kapitanski, A. Kelkar, and W. White (2000). Matching, linear systems, and the ball and beam. *Preprint.*

Apkarian, J. (1994). Control System Laboratory, Quanser Consulting, Hamilton, Ontario, Canada L8R 3K8.

Auckly, D., L. Kapitanski, and W. White (2000). Control of nonlinear underactuated systems. To appear in *Commun. Pure Appl. Math.*

Auckly, D., L. Kapitanski (2000). Mathematical Problems in the Control of Underactuated Systems. *Preprint.*

Bloch, A., N. Leonard and J. Marsden (1998). Matching and stabilization by the method of controlled Lagrangians. *Proc. IEEE Conf. on Decision and Control*, Tampa, FL, pp. 1446-1451.

Bloch, A., N. Leonard and J. Marsden (1999). Stabilization of the pendulum on a rotor arm by the method of controlled Lagrangians. *Proc. IEEE Int. Conf. on Robotics and Automation*, Detroit, MI, pp. 500-505.

Hamberg, J.(1999). General matching conditions in the theory of controlled Lagrangians. *Proceedings of the 38th Conference on Decision and Control*, Phoenix, AZ.

van der Schaft, A. J. (1986). Stabilization of Hamiltonian systems. *Nonlinear Analysis, Theory, Methods & Applications*, **10**, 1021-1035.